On a Non Linear Geometrical Inverse Problem of Signorini type: Identifiability and Stability

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On a non linear geometrical inverse problem of Signorini type : identifiability and stability

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Abstract: This report deals with a non linear inverse problem : identification of unknown boundaries, on which the prescribed conditions are of Signorini type. We first prove an identifiability result, in both frameworks of steady state thermal and elastostatics testing. Local Lipschitz stability of the solutions, with respect to the boundary measurements, is also established under the assumption that the unknown boundary is part of a $\mathcal{C}^{1,\beta}$ Jordan curve, with $\beta > 0$.

Key-words: geometrical inverse problems, identification, Signorini type boundary conditions, unknown boundaries, identifiability, Lipschitz local stability, domain derivatives, optimal shape design.

(Résumé : tsvp)

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Sur un problème inverse géométrique non linéaire de type Signorini

Résumé : On s'intéresse dans ce rapport à un problème inverse non linéaire d'identification de frontières inconnues par des mesures de surface, les conditions aux limites étant de type Signorini. On montre d'abord un résultat d'identifiabilité, valable pour les mesures thermiques dans un cadre stationnaire, comme pour les mesures élastiques, dans un cadre linéaire, homogène et isotrope. La stabité locale lipschitzienne des solutions vis à vis des mesures est ensuite prouvée, sous l'hypothèse que la frontière inconnue est une partie de courbe de Jordan de classe $\mathcal{C}^{1,\beta}$, avec $\beta > 0$.

Mots-clé : problèmes inverses géométriques, identification, conditions aux limites de Signorini, détection de frontières, identifiabilité, stabilité locale lipschitzienne, dérivation par rapport au domaine, optimisation de forme.

1 Introduction

This work is devoted to the study of an inverse geometrical problem, which consists in finding the shape of an unknown part γ of the boundary $\partial\Omega$ of a two-dimensional body Ω. The two extremal points of the unknown boundary γ are supposed to be known, and boundary conditions of Signorini type are prescribed on γ. In the elasticity framework, the direct problem modelizes states of equilibrium of a linear elastic body, the part γ of its boundary being supported by a non deformable friction-free surface.

The practical motivation of this work is related to non destructive control processes. Using steady thermal, electrical, or elastic measurements, the governing state equation (or system) is elliptic (Laplace equation, or Lamé system. Our interest is focused on uniqueness and stability questions. Uniqueness is a crucial point in this kind of problems, since it informs us us if a single measurement (or a finite number of them) is enough to insure the identifiability.

Many theoretical studies have been performed for the similar problem of conductivities identification. Kohn & Vogelius [14] established first in 1985 the uniqueness, with infinitely many measurements (that is the whole Neumann- Dirichlet operator), for inclusion domains with analytical boundaries, while Isakov [10, 1988] proved later the same result for Lipschitz boundaries. But the most interesting results, for practical purposes, will come later on, when uniqueness is proved for a single measurement, or at least for a finite number of them. Bellout & Friedman [4, 1988], Alessandrini ([1, 1988]), Isakov [10, 1988], Friedman & Isakov [8, 1989], Isakov & Powell [11, 1990], as well as Bellout, Friedman & Isakov [5, 1992] proved uniqueness results, dropping or weakening paper after paper some of the restrictive assumptions (regularity, monotonicity,...) made previously on the admissible boundaries. As for the papers involving Bellout & Friedman ([4] , [5]), the identifiability results are obtained as consequences of Lipschitz local stability ones. Such a question is also present in all Alessandrini's papers. For linear and non linear boundary conditions, uniqueness local Lipshitz stability (in the linear case) results have also been established in [2, 1993] and [3, 1997].

Section 2 is devoted to uniqueness (*identifiability*) questions, in the thermal framework, as well as in the elasticity framework. In the third section, we deal with stability questions. A local Lipshitz stability result is proved, under the assumption that the boundary γ be part of some $\mathcal{C}^{1,\beta}$ Jordan curve for some $\beta > 0$, by using domain derivative techniques, as well as arguments related to analytical functions theory.

2 Identifiability

Let Ω denote a 2D or 3D domain occupied by the body, and $\partial\Omega$ its boundary, that we shall divide in three parts as shown in Figure 1 :

$$\partial\Omega = \gamma \cup \Gamma_D \cup \Gamma_N$$

where γ is the unknown part, Γ_N the part where the fluxes used for the measurements are prescribed, and Γ_D the part where an homogeneous Dirichlet condition is prescribed in order to get a well-posed direct problem.

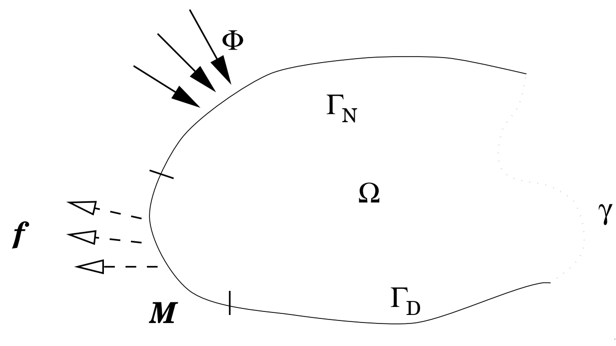

Figure 1: *The domain and its boundary*

2.1 Case of steady state thermal testing

2.1.1 The direct problem.

Let us denote by Ω_γ the domain Ω with unknown boundary γ. The direct problem is therefore given by :

$$\left\{\begin{array}{rcll} \Delta u_\gamma & = & 0 & \text{in} \quad \Omega_\gamma \\ u_\gamma & = & 0 & on \quad \Gamma_D \\ \frac{\partial u_\gamma}{\partial n} & = & \phi & on \quad \Gamma_N \\ u_\gamma \geq 0 & \frac{\partial u_\gamma}{\partial n} \geq 0 & u_\gamma \frac{\partial u_\gamma}{\partial n} = 0 & on \quad \gamma \end{array}\right. \tag{1}$$

where ϕ is a prescribed heat flux on Γ_N ; $\phi \in H^{-\frac{1}{2}}(\Gamma_N)$ and $\phi \not\equiv 0$ on Γ_N.

The associated variational formulation of such an elliptic inequation, as well as the existence and uniqueness of the solution, are well known (see for example [9, 1976]). Let us briefly recall two equivalent formulations of problem (1) :

$$\left\{\begin{array}{lcl} u_\gamma \in K & & \\ a(u_\gamma, v - u_\gamma) & \geq & L(v - u_\gamma) \quad \forall v \in K \end{array}\right. \tag{2}$$

or

$$\left\{\begin{array}{lcl} u_\gamma \in K & & \\ J(u_\gamma) & \leq & J(v) \quad \forall v \in K \end{array}\right. \tag{3}$$

where K is the closed convex set of $H^1(\Omega_\gamma)$ defined by :

$$K \quad = \quad \{v \in H^1(\Omega_\gamma)\,;\ v = 0 \text{ on } \Gamma_D \text{ and } v \geq 0 \text{ on } \gamma\}$$

and where, for u and v in $H^1(\Omega_\gamma)$:

$$\left\{\begin{array}{rcl} a(u,v) & = & \int_{\Omega_\gamma} \nabla u \,.\, \nabla v \\ L(v) & = & \int_{\Gamma_N} \phi\, v \\ J(v) & = & \frac{1}{2}\, a(v,v) - L(v) \end{array}\right.$$

2.1.2 Uniqueness for the inverse problem.

Let M be an open subset, with positive measure, of Γ_N. We are going to prove the identifiability result, which is that two different admissible boundaries γ_1 and γ_2 cannot produce the same measured temperature f on M for the given flux ϕ.

Theorem 1 (identifiability) *Let γ_1 and γ_2 be two piecewise $\mathcal{C}^{1,1}$ boundaries having the same extremity points. Assume that the corresponding domains $\Omega_i = (\Omega_{\gamma_i})$ are connected, and let $u_i = u_{\gamma_i}$ be the solution of problem (1) in domain $\Omega_{\gamma_i} = \Omega_i \quad (i = 1, 2)$. Then, if $u_1|_M = u_2|_M$, the boundaries γ_1 and γ_2 coincide.*

Proof : Let us denote by Ω_{12} the intersection $\Omega_{\gamma_1} \cap \Omega_{\gamma_2}$ (see figure2), and let $\omega = u_1 - u_2$. ω is then solution of the Cauchy problem :

$$\left\{ \begin{array}{rcll} \Delta\omega & = & 0 & \text{in} \quad \Omega_{12} \\ \omega & = & 0 & \text{on} \quad M \\ \frac{\partial}{\partial n}\omega & = & 0 & \text{on} \quad M \end{array} \right.$$

Since Ω_{12} is connected, ω vanishes on the whole domain Ω_{12} by the Holmgren's unique continuation theorem. Thus :

$$u_1 = u_2 \quad \text{on} \quad \partial(\Omega_1 \cap \Omega_2) \tag{4}$$

and also :

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \quad \text{on} \quad \partial(\Omega_1 \cap \Omega_2) \tag{5}$$

Let $\mathcal{O} = (\Omega_1 \cup \Omega_2) \backslash \overline{(\Omega_1 \cap \Omega_2)}$. Assume that $\mathcal{O} \neq \emptyset$, and let $\mathcal{O}_1$ be one of its connected components. Assume for instance that $\mathcal{O}_1 \subset \Omega_1 \backslash \Omega_2$.

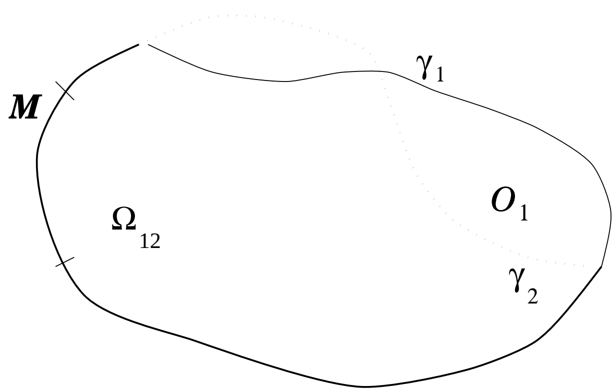

Figure 2: *Two possible boundaries*

Then, in the open set $\mathcal{O}_1$, u_1 is solution of :

$$\left\{ \begin{array}{rcll} \Delta u_1 & = & 0 & \text{in} \quad \mathcal{O}_1 \\ u_1 \frac{\partial u_1}{\partial n} & = & 0 & \text{on} \quad \partial\mathcal{O}_1 \cap \gamma_1 \\ u_1 \frac{\partial u_1}{\partial n} & = & 0 & \text{on} \quad \partial\mathcal{O}_1 \cap \gamma_2 \end{array} \right.$$

The boundary conditions on $\partial\mathcal{O}_1 \cap \gamma_1$ and $\partial\mathcal{O}_1 \cap \gamma_2$ come from the Signorini boundary condition, and from (4) and (5). It comes then that :

$$\int_{\mathcal{O}_1} |\nabla u_1|^2 = \int_{\partial\mathcal{O}_1} u_1 \frac{\partial u_1}{\partial n} = 0$$

and thus u_1 is constant in $\mathcal{O}_1$. By analyticity, u_1 is constant in the whole domain Ω_1, and thus $\frac{\partial u_1}{\partial n} = 0$ on $\partial\Omega_1$, which is in contradiction with $\phi \not\equiv 0$ on Γ_N.

■

Remark 1 : This proof drops the interior sphere assumption on the domain Ω_γ, which was needed in [2] in order to use the Hopf maximum principle.
Remark 2 : This result extends to any analytical hypo-elliptic operator, for example for any elliptic operator with constant coefficients.

2.2 Case of elastostatics testing

Let us denote by u the displacement field, ε the associated linearized strain tensor, and σ the stress tensor. The material is supposed to be isotropic and homogeneous, and the constitutive law is linear. The stiffness tensor R then fulfills the classical symmetry and ellipticity conditions, that is for some real positive number ρ :

$$\left\{ \begin{array}{l} R_{ijkl} = R_{jikl} = R_{klij} \quad i,j,k,l = 1,2 \\ \displaystyle\sum_{k,l=1,2} R_{ijkl}\,\xi_{ij}\,\xi_{kl} \geq \rho \sum_{i,j=1}^{2} (\xi_{ij})^2 \quad \forall \xi \in \mathbf{R}^4 \end{array} \right. \tag{6}$$

The direct problem is then the following :

$$\left\{ \begin{array}{rcll} \sigma_{ij,i} & = & 0 & \text{in } \Omega_\gamma \\ \sigma_{ij} & = & R_{ijkl}\,(\varepsilon_{kl}(u)) & \text{in } \Omega_\gamma \\ u & = & 0 & on\ \Gamma_D \\ \sigma_{ij}\,n_j & = & g_i & on\ \Gamma_N \\ \sigma_{ij}\,t_j & = & 0 & on\ \gamma \\ (\sigma_{ij} n_i)\,n_j \leq 0 & u\,.\,n \leq 0 & ((\sigma\,.\,n)\,.\,n)\,(u\,.\,n) = 0 & on\ \gamma \end{array} \right. \tag{7}$$

where g is a prescribed load on Γ_N ($g \in (H^{-\frac{1}{2}}(\Gamma_N))^2$ and $g \not\equiv 0$ on Γ_N).

It is well known that the solution of (7) is unique (see for example [13, 1988]), and the associated variational formulations are similar to (2) and (3), where the convex set K is defined as follows :

$$K = \{v \in (H^1(\Omega_\gamma))^2 \,;\, v = 0 \text{ on } \Gamma_D \text{ and } v.n \leq 0 \text{ on } \gamma\}$$

We can then settle the identifiability result exactly in the same way than for thermal testing (theorem 1). Its proof works also the same way, except it uses Almansi's lemma, which generalises Holmgren's theorem to elliptic systems [17].

Remark : Although they were formulated in 2D situations, these identifiability results extend without difficulty to 3D.

3 Stability.

In this section, problem (1) is again considered. The overspecified data on the open set M of the boundary $\partial\Omega$ have been obtained by measurements, and are thus subject to errors. The stability means, roughly speaking, that *small* errors on the measurements lead to *small* perturbations on the unknown geometry. To formalize this idea, let us consider a set Γ_{ad} of admissible geometries, and the mapping η defined, the *identifying* flux ϕ of the previous section being given, by :

$$\begin{array}{rcl} \eta \ : \ \Gamma_{ad} & \longmapsto & L^2(M) \\ \gamma & \longmapsto & f = u_\gamma|_M \end{array}$$

The identifiability result proved in the previous section means that this mapping is one-to-one, and therefore, that the mapping :

$$\begin{array}{rcl} \eta \ : \ \Gamma_{ad} & \longmapsto & \eta(\Gamma_{ad}) \\ \gamma & \longmapsto & f = u_\gamma|_M \end{array}$$

is invertible. The stability will be established if one proves, after having equipped Γ_{ad} with an appropriate topology, that η^{-1} is continuous. But this might be not sufficient for numerical purposes. This is the reason why we shall be focusing our attention on Lipschitz stability, even if the results expected hold only locally. We shall be using for that the derivatives with respect to the domain as a basic tool.

3.1 Derivatives of the solution with respect to the domain

To prove local stability results, we need to map an admissible boundary onto another one, close to it. Following Murat-Simon [16], we shall use mappings from the whole domain Ω onto Ω_h, defined as follows :

$$F_h = Id + h\theta$$

where θ is a $\left(\mathcal{C}^1(\mathcal{B})\right)^2$ vector-field defined on some bounded ball $\mathcal{B}$ containing $\overline{\Omega_\gamma}$, verifying :

$$\left\{ \begin{array}{rcl} \theta & \equiv & 0 \quad \text{on } \Gamma_D \cup \Gamma_N \\ \theta & \equiv & 0 \quad \text{in some neighbourhood of } \Gamma_D \\ \theta\,.\,\tau & = & 0 \quad \text{on } \gamma \\ \theta_n \ = \ \dfrac{\partial \theta_n}{\partial \tau} & = & 0 \quad \text{on } \partial\gamma \end{array} \right. \tag{8}$$

There exists some constant h_0 such that F_h be a diffeomorphism for any h such that $|h| < h_0$. Let us denote by Ω_{γ_h}, or by Ω_h the set :

$$\Omega_h = (Id + h\,\theta)\,(\Omega) \tag{9}$$

the boundary of which is $\partial\Omega_h = \gamma_h \cup \Gamma_D \cup \Gamma_N$, γ_h being the image by F_h of γ. Let us now denote by u_h the solution of the non linear boundary problem on Ω_h :

$$\left\{ \begin{array}{lcll} \Delta u_h & = & 0 & \text{in} \quad \Omega_h \\ u_h & = & 0 & \text{on} \quad \Gamma_D \\ \frac{\partial u_h}{\partial n} & = & \phi & \text{on} \quad \Gamma_N \\ u_h \geq 0 & \dfrac{\partial u_h}{\partial n} \geq 0 & u_h \dfrac{\partial u_h}{\partial n} = 0 & \text{on} \quad \gamma_h \end{array} \right. \tag{10}$$

and let u^h be its "transported" on the original domain Ω_γ, also denoted Ω :

$$u^h = u_h \circ F_h \tag{11}$$

Denoting by u^0 the solution u_γ of problem (1), we can define a partition of the unknown boundary γ into two parts : a "Dirichlet" part γ_D on which the boundary condition $u^0 = 0$ is fulfilled, and a "Neumann" part $\gamma_N = \gamma \setminus \gamma_D$. Then :

$$\gamma_D = \left\{ x \in \gamma \,;\, u^0(x) = 0 \right\} \tag{12}$$

and therefore :

$$\gamma_N = \gamma \setminus \gamma_D = \left\{ x \in \gamma \,;\, u^0(x) > 0 \right\} \tag{13}$$

Let us then define the following convex set :

$$\mathcal{S} = \left\{ v \in H^1(\Omega_\gamma)\,;\, \int_\Omega \nabla u^0 \, \nabla v = \int_{\Gamma_N} \phi\, v\,;\, v|_{\Gamma_D} = 0 \text{ and } v \geq 0 \text{ a.e. on } \gamma_D \right\} \tag{14}$$

The following expansion result is due to J. Sokolowski & J.P. Zolesio [20, 1982].

Theorem 2 (Sokolowski-Zolesio) *The scalar field u^h can be expanded as follows :*

$$u^h = u^0 + h\, u^1 + h\, O(h) \tag{15}$$

where u^1 and $O(h)$ are elements of $H^1(\Omega)$ verifying :

- $\lim_{h \to 0} O(h) = 0$ *in* $H^1(\Omega)$
- u^1 *is the unique solution of the following variational inequality in $\mathcal{S}$:*

$$\int_\Omega \nabla u^1 \, \nabla v \geq \int_\Omega \left(\frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M} \right) \nabla u^0 \, \nabla v - \int_\Omega \left(\nabla u^0 \, \nabla v \right) \, div\, \theta \tag{16}$$

for any $v \in \mathcal{S}$

3.2 Local Lipshitz stability

From now on, we shall denote by a subscript N the solutions of problems with a prescribed Neumann boundary condition (i.e. the given flux $\dfrac{\partial u}{\partial n} = \phi$ on Γ_N), and by the subscript D the solutions of problems with the measured temperature as prescribed Dirichlet boundary value (i.e. $u|_M = f$). The solution of the Signorini direct problem (1), with the prescribed flux ϕ, will then be denoted u_N, or u_N^0, and its derivative with respect to the domain u_N^1. The solution of the Signorini problem with prescribed flux on the perturbated domain Ω_h will be denoted u_{hN}, while its "transported" on the original domain Ω is u_N^h.

At this point, we have also to make it clear that we denote by γ the unknown boundary, *without its extreme points*, so that it is an open subset of $\partial\Omega$.

3.2.1 Some preliminary technical results

The proof of the local Lipschitz stability result is somewhat technical, and needs some additional light to be thrown on the topological features of the partition $(\gamma_D\,,\, \gamma_N)$ of the unknown boundary γ. The desired result would be that the sets γ_D and γ_N defined by (12) and (13), be also characterized - up to neglectible sets - as follows :

$$\gamma_D = \left\{ x \in \gamma\,;\, \frac{\partial u_N^0}{\partial n}(x) > 0 \right\} \text{ and thus } \gamma_N = \left\{ x \in \gamma\,;\, \frac{\partial u_N^0}{\partial n}(x) = 0 \right\}$$

Actually, such a result could not be proved. The reason is that the set $(\gamma_D \setminus \overset{\circ}{\gamma}_D)$, where both u_N^0 and $\dfrac{\partial u_N^0}{\partial n}$ could vanish, might be some closed subset of γ_D of positive measure and void interior, such as Cantor p-adic sets. As far as such possibility is not excluded, the best we can expect in characterizing these sets is the following.

Theorem 3 *Assume the unknown boundary γ be part of a $\mathcal{C}^{1,\beta}$ Jordan curve, for some $\beta > 0$. Then, γ_N is an open subset of γ, and γ_D a closed one, the interior of which is - up to a neglectible set - the following :*

$$\overset{\circ}{\gamma}_D = \left\{ x \in \gamma \,;\, \frac{\partial u_N^0}{\partial n}(x) > 0 \right\} \tag{17}$$

Proof : As a matter of fact, it happens that the solution u_N^0 of the Signorini problem is locally (i.e. in the vicinity of any open subset of γ) smoother than the solution of the associated mixed Dirichlet-Neumann problem (Dirichlet boundary condition on γ_D, Neumann boundary condition on γ_N) : singularities having a $\rho^{\frac{1}{2}}$-behaviour at the vicinity of the switch-points (ρ being the distance to these points) from one boundary to another occur for this latter, while the positiveness condition eliminates them from the Signorini solution. More precisely, according to Lions [15, 1969] and to Khodja-Moussaoui [12, 1992], we have :

$$u_N^0 \in H^2(\mathcal{O}) \cap \mathcal{C}^{1,\beta}(\overline{\mathcal{O}}), \text{ with } 0 < \beta < \frac{1}{2} \tag{18}$$

$\mathcal{O}$ being any open subset of Ω such that $\overline{\mathcal{O}} \cap \partial\Omega \subset \gamma$.

It follows that u_N^0 is continuous on γ, so that γ_N is an open subset, and γ_D a closed one of γ. $\dfrac{\partial u_N^0}{\partial n}(x)$ is also continuous on γ, so that by the Signorini condition, $\dfrac{\partial u_N^0}{\partial n}(x) = 0$ on γ_N. The set :

$$\mathcal{A} := \left\{ x \in \gamma \,;\, \frac{\partial u_N^0}{\partial n}(x) > 0 \right\}$$

is therefore an open subset of γ_D, and accordingly of its interior $\overset{\circ}{\gamma}_D$.

To prove that $\mathcal{A} = \overset{\circ}{\gamma}_D$, it is sufficient to establish that the set $(\overset{\circ}{\gamma}_D \setminus \mathcal{A})$ has no accumulation points in $\overset{\circ}{\gamma}_D$, which would insure that its measure is zero. The forthcoming lemma is dealing with this issue.

$\blacksquare$

Lemma 1 *The set $(\overset{\circ}{\gamma}_D \setminus \mathcal{A}) = \{ x \in \overset{\circ}{\gamma}_D \,;\, \nabla u(x) = 0 \}$ has no accumulation points in $\overset{\circ}{\gamma}_D$, which means that all its points are isolated.*

Proof of the lemma : Let x_0 be a non-isolated point of $(\overset{\circ}{\gamma}_D \setminus \mathcal{A})$. Let ϑ_{x_0} be some open neighbourhood of x_0, included in $\overset{\circ}{\gamma}_D$. We can find some sequence $x_n \in (\overset{\circ}{\gamma}_D \setminus \mathcal{A}) \cap \vartheta_{x_0}$ verifying :

$$\left\{ \begin{array}{l} x_n \neq x_0 \, \forall\, n \in \mathbf{N} \\ \lim\limits_{n \to +\infty} x_n = x_0 \end{array} \right. \tag{19}$$

Choose now some open subset $\mathcal{O}$ of Ω such that $\overline{\mathcal{O}} \cap \partial\Omega \subset \vartheta_{x_0}$. $\partial\mathcal{O}$ is part of some $\mathcal{C}^{1,\beta}$ Jordan curve, and it is then possible to find two conformal mappings φ and ψ, the first one mapping the unit disc $\mathcal{D}$ on $\mathcal{O}$ and the second one mapping $\mathcal{D}$ on some simply connected domain Θ of the half plane $\{(x,y) \in \mathbf{R}^2 \,;\, y \geq 0\}$. The boundary of Θ is some $\mathcal{C}^{1,\beta}$ Jordan curve, and we can suppose a part of it is a segment [a,b] included in the x-axis.
Moreover, by the Kellogg-Warschawski theorem (see Pommerenke [18]), we have :

$$\left\{ \begin{array}{l} \varphi \text{ (and } \psi) \text{ are diffeomorphisms from } \overline{\mathcal{D}} \text{ on } \overline{\Omega} \text{ (and on } \overline{\Theta}) \\ \varphi \text{ and } \psi \text{ are differentiable on } \overline{\mathcal{D}} \\ \varphi'(z) \neq 0 \text{ and } \psi'(z) \neq 0 \quad \forall\, z \in \overline{\mathcal{D}} \end{array} \right. \tag{20}$$

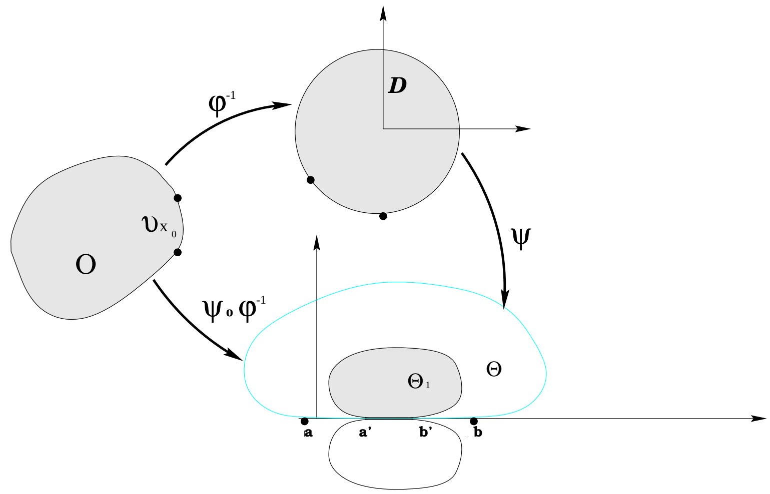

Figure 3: **The conformal mappings**

The function $w = u_N^0 \circ (\varphi \circ \psi^{-1})$ is then harmonic in Θ. $x_0 \in \overset{\circ}{\gamma}_D$ and , up to a rotation, we can assume that $(\psi \circ \varphi^{-1})(x_0) \in]a, b[\times\{0\}$. Since u_N^0 vanishes in some neighbourhood of x_0, w vanishes in the associated neighbourhood of $(\psi \circ \varphi^{-1})(x_0)$, and we can then find two real numbers a' and b', $a < a' < b' < b$ such that $w(t, 0) = 0 \quad \forall\, t \in]a', b'[$.

Let now Θ_1 be an open subset of Θ with $\mathcal{C}^1$ boundary, such that $\partial\Theta_1 \cap (x'ox) = [a', b'] \times \{0\}$. Let us denote by Θ_1^s the symmetrized set, with respect to the x-axis, of Θ_1. This set includes $]a', b'[\times 0$, and we can define on it the harmonic function $\tilde{w}$, by theSchwarz reflexion principle :

$$\begin{cases} \tilde{w}(x, y) = w(x, y) \text{ if } y \geq 0 \\ \tilde{w}(x, y) = -\, w(x, -y) \text{ if } y < 0 \end{cases} \tag{21}$$

There exists some integer N such that $\forall n \geq N$, $\psi \circ \varphi^{-1}(x_n) \in]a', b'[\times\{0\}$. Let t_n be the point of]a',b'[such that :

$$\psi \circ \varphi^{-1}(x_n) = (t_n, 0) \quad \forall n \geq N$$

Then :

$$\begin{cases} \lim\limits_{n \to +\infty} (t_n, 0) = \psi \circ \varphi^{-1}(x_0) \\ \nabla w(t_n, 0) = 0 \quad \forall\, n \geq N \\ (t_n\, ,\, 0) \neq \psi \circ \varphi^{-1}(x_0) \end{cases} \tag{22}$$

$\tilde{w}$ is the real part of an holomorphic function h in Θ_1^s. According to the Cauchy-Riemann conditions, the imaginary part's gradient of such a function will also vanish in $(t_n, 0)$ for all $n \geq N$. This means that, inside the domain Θ_1^s, the zeros of $h'(z)$ are not isolated, which is not possible since h' is holomorphic in this domain.

■

The forthcoming lemmas are technical points needed for the proof of the final stability result. Proofs of lemmas 2 and 5 are not really different from the linear case. Their statements are recalled mostly for the reader's convenience.

Let us define the $L^2(\Omega)$ function $u'_N = u_N^1 - \langle \nabla u_N^0 \,,\, \theta \rangle$. This function is known as the *Eulerian* derivative of the solution with respect to the domain, while u_N^1 is the *Lagrangian* one. The first result is that $\Delta u'_N = 0$ in Ω.

Lemma 2 *Define the L^2 function $u'_N = u^1_N - \langle \nabla u^0_N , \theta \rangle$. Then :*

$$\Delta u'_N = 0 \text{ in } \Omega$$

Proof : It works exactly as for the linear case [19, Simon].

■

Lemma 3 *Suppose u^1_N vanishes on M. Then, $u^1_N = \langle \nabla u^0_N , \theta \rangle$ in Ω, and moreover :*

$$\beta := \left\langle \frac{\partial u^0_N}{\partial n} , \theta_n \frac{\partial u^0_N}{\partial n} \right\rangle_{H^{-\frac{1}{2}}(\overset{\circ}{\gamma}_D) \times H^{\frac{1}{2}}(\overset{\circ}{\gamma}_D)} = 0 \tag{23}$$

Proof : Let us choose some $v \in H^1(\Omega)$ such that $v|_{\gamma \cup \Gamma_D} = 0$. Therefore, $v \in \mathcal{S}$ and also $-v \in \mathcal{S}$, so that inequation (16) leads to the following equation :

$$\int_\Omega \nabla u^1_N \nabla v = \int_\Omega \left(\frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M} \right) \nabla u^0_N \nabla v - \int_\Omega \left(\nabla u^0_N \nabla v \right) \text{ div } \theta \tag{24}$$

which gives, by Green formula :

$$\frac{\partial u^1_N}{\partial n} = \left\langle \left(\frac{\partial \theta}{\partial M} + \frac{\partial \theta}{\partial M}^t \right) \nabla u^0_N , \vec{n} \right\rangle - \text{div}\theta \, \frac{\partial u^0_N}{\partial n} \text{ on } M$$

θ vanishes in a neighbourhood of M, and thus $\dfrac{\partial u^1_N}{\partial n} = 0$ on M. u'_N is hence solution of :

$$\begin{cases} \Delta u'_N = 0 & \text{in } \Omega \\ u'_N = 0 & \text{on } M \\ \dfrac{\partial u'_N}{\partial n} = 0 & \text{on } M \end{cases} \tag{25}$$

By the Holmgren's unique continuation theorem, $u'_N = 0$ in Ω, and it follows that :

$$u^1_N = \langle \nabla u^0_N , \theta \rangle \tag{26}$$

By the above identity, the trace of u^1_N on $\partial\Omega$ is $\theta_n \dfrac{\partial u^0_N}{\partial n}$, which is therefore an element of $H^{\frac{1}{2}}(\partial\Omega)$. On the other hand, $u^0_N \in \{ v \in H^1(\Omega) \, ; \, \Delta v \in L^2(\Omega) \}$, so that we can define its normal derivative as an element of $H^{-\frac{1}{2}}(\partial\Omega)$, and accordingly as an element of $H^{-\frac{1}{2}}(\overset{\circ}{\gamma}_D)$. The duality product in (23) makes then sense.

u^1_N vanishes wherever θ does, and in particular :

$$u^1_N = 0 \text{ on } \Gamma_N \cup \Gamma_D$$

Since $u^1_N \in \mathcal{S}$, we have $\displaystyle\int_\Omega \nabla u^0_N \nabla u^1_N = \int_{\Gamma_N} \phi \, u^1_N = 0$ and, by using Green formula and theorem 3, we get (23).

■

Lemma 4 *Suppose* $\overset{\circ}{\gamma}_D = \emptyset$. *Then,* $meas(\gamma_D) = 0$.

Proof : Suppose γ_D has an accumulation point x_0, interior to γ. There exists then some open "interval" ϑ_{x_0} of γ, containing x_0, and a sequence of points $x_n \in \gamma_D \cap \vartheta_{x_0}$, each one of them being supposed to be different from the other, such that :

$$\left\{ \begin{array}{ccc} x_n & \neq & x_0 \quad \forall n \in \mathbf{N} \\ \lim\limits_{n\to+\infty}(x_n) & = & x_0 \\ u_N^0(x_n) & = & 0 \quad \forall n \in \mathbf{N} \end{array} \right. \tag{27}$$

By Khodja-Moussaoui [12], we know that $u_N^0 \in \mathcal{C}^{1,\beta}(\vartheta_{x_0})$. It comes out that we can find a sequence $\xi_n \in]x_n, x_{n+1}[$, such that $\dfrac{\partial u_N^0}{\partial \tau}(\xi_n) = 0$.

The sequence ξ_n is not stationary, and converges to x_0. Furthermore, $\dfrac{\partial u_N^0}{\partial n}(\xi_n) = 0 \quad \forall\, n$, since $\dfrac{\partial u_N^0}{\partial n} = 0$ on γ, so that all points ξ_n are in the set $\{x \in \vartheta_{x_0} \,;\, \nabla u_N^0(x) = 0\}$.

To conclude, we shall apply the result proved in lemma 1 to the conjugate function of u_N^0 : Let $h = u_N^0 + iv$ be the holomorphic function associated with u_N^0. By the Cauchy-Riemann conditions, $\dfrac{\partial v}{\partial \tau} = 0$ on ϑ_{x_0}, so that the function $w = v - v(x_0)$ is a harmonic function vanishing on ϑ_{x_0}. Therefore, x_0 is an accumulation point of the set $\{x \in \vartheta_{x_0} \,;\, \nabla w(x) = 0\}$, which by lemma 1 cannot have any. This ends the proof of lemma 4.

■

Lemma 5 *Let ϑ_{x_0} be some open subset of γ, and $\mathcal{O}$ some open subset of Ω such that $\overline{\mathcal{O}} \cap \partial\Omega \subset \vartheta_{x_0}$. Then, for any $v \in H^2(\mathcal{O})$, vanishing in some neighbourhood of $\partial\mathcal{O} \setminus \vartheta_{x_0}$, we have :*

$$\int_{\vartheta_{x_0}} \theta_n \frac{\partial u_N^0}{\partial \tau} \frac{\partial v}{\partial \tau} = 0 \tag{28}$$

Proof : Let v be such a test function.

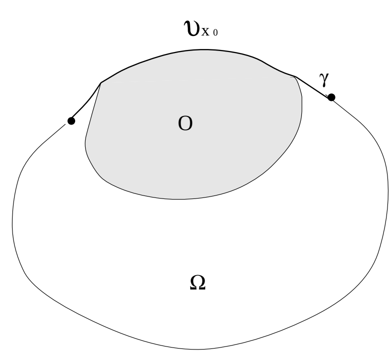

Figure 4: **The open set O**

Then, $v \in \mathcal{S}$ and $(-v) \in \mathcal{S}$, so that inequality (16) becomes the equality :

$$\int_\Omega \nabla u_N^1 \nabla v = \int_\Omega \left(\frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M}\right) \nabla u_N^0 \nabla v - \int_\Omega \langle \nabla u_N^0 \,,\, \nabla v \rangle \; \mathrm{div}\theta \tag{29}$$

By Green formula, the left handside of the above equation becomes :

$$\int_\Omega \nabla u_N^1 \,\nabla v = -\int_{\mathcal{O}} u_N^1 \,\Delta v + \int_{\partial\mathcal{O}} u_N^1 \,\frac{\partial v}{\partial n}$$

The proof ends therefore exactly as for the linear case, by using Green formulae and lemma 3 [2].

■

3.2.2 The final stability result

We are now able to prove the final Lipschitz stability result.

Theorem 4 (Lipschitz stability) *Suppose θ fulfills (8) and $\theta_n \not\equiv 0$ on γ. Then, denoting $u_{hN}|_M$ by f_h, we have :*

$$\lim_{h\to 0} \frac{|f - f_h|_{L^2(M)}}{h} > 0 \tag{30}$$

Proof : According to the expansion (15), (30) is equivalent to the following :

$$\left|u_N^1\right|_{L^2(M)} > 0 \tag{31}$$

Let us suppose that $u_N^1 = 0$ a.e. on M. We shall now consider two cases :

• **First case** : $\overset{\circ}{\gamma}_D = \emptyset$

By theorem 3, we derive that $\dfrac{\partial u_N^0}{\partial n} = 0$ on γ, and u_N^0 is then solution of the linear "Neumann" problem :

$$\left\{\begin{array}{rcll} \Delta u_N^0 & = & 0 & \text{in} \quad \Omega_\gamma \\ u_N^0 & = & 0 & \text{on} \quad \Gamma_D \\ \dfrac{\partial u_N^0}{\partial n} & = & \phi & \text{on} \quad \Gamma_N \\ \dfrac{\partial u_N^0}{\partial n} & = & 0 & \text{on} \quad \gamma \end{array}\right. \tag{32}$$

On the other hand, we also know that u_N^1 is solution of :

$$\left\{\begin{array}{rcl} u_N^1 & \in & \mathcal{S} \\ \displaystyle\int_\Omega \nabla u_N^1 \,\nabla v & \geq & \displaystyle\int_\Omega \left(\frac{\partial\theta}{\partial M}^t + \frac{\partial\theta}{\partial M}\right) \nabla u^0 \,\nabla v - \int_\Omega \left(\nabla u^0 \,\nabla v\right) \,\mathrm{div}\theta \;,\; \forall\, v \in \mathcal{S} \end{array}\right. \tag{33}$$

Since $\mathrm{meas}(\gamma_D) = 0$ (by lemma 4), it comes that $\mathcal{S} = V := \left\{v \in H^1(\Omega)\,;\, v = 0 \text{ on } \Gamma_D\right\}$. u_N^1 is therefore solution of the linear problem :

$$\left\{\begin{array}{rcl} u_N^1 & \in & V \\ \displaystyle\int_\Omega \nabla u_N^1 \,\nabla v & = & \displaystyle\int_\Omega \left(\frac{\partial\theta}{\partial M}^t + \frac{\partial\theta}{\partial M}\right) \nabla u_N^0 \,\nabla v - \int_\Omega \left(\nabla u_N^0 \,\nabla v\right) \,\mathrm{div}\;\theta \;,\; \forall\, v \in V \end{array}\right. \tag{34}$$

Actually, this means that u_N^1 is the derivative, with respect to the domain, of the linear "Neumann" problem. Referring to [2], we know that $\left|u_N^1\right|_{L^2(M)} > 0$, which is is in contradiction with our assumption on $u_N^1|_M = 0$. This ends the proof for this first case.

• **Second case** : $\overset{\circ}{\gamma}_D \neq \emptyset$

By theorem 3, $\dfrac{\partial u_N^0}{\partial n}$ is a strictly positive distribution on $\overset{\circ}{\gamma}_D$, and $\theta_n \dfrac{\partial u_N^0}{\partial n} = u_N^1$ is positive on $\overset{\circ}{\gamma}_D$, since $u_N^1 \in \mathcal{S}$.

- If $\overset{\circ}{\gamma}_D = \gamma$, the proof ends here since this is in contradiction with the assumption $\theta_n \not\equiv 0$ on γ.
- Let us therefore consider the case $\overset{\circ}{\gamma}_D \neq \gamma$.

By lemma 3, we know that $\beta := \left\langle \dfrac{\partial u_N^0}{\partial n}, \theta_n \dfrac{\partial u_N^0}{\partial n} \right\rangle_{H^{-\frac{1}{2}}(\overset{\circ}{\gamma}_D) \times H^{\frac{1}{2}}(\overset{\circ}{\gamma}_D)} = 0$, and we can then conclude that $\theta_n \dfrac{\partial u_N^0}{\partial n} = 0$ on $\overset{\circ}{\gamma}_D$, which by using the characterization of theorem 3 gives $\theta_n \equiv 0$ on $\overset{\circ}{\gamma}_D$.

There exists some point x_0 of γ_N where θ_n does not vanish, and therefore some open connected neighbourhood ϑ_{x_0} of x_0 in γ_N where θ_n does not change sign. Otherwise, assuming θ_n vanishes on γ_N, and since it does not vanish identically on γ, there would exist some point $x_0 \in \gamma$ such that $\theta_n(x_0) \neq 0$. θ_n being continuous - it would not vanish in some open neighbourhood ϑ_{x_0} of x_0. Of course, $\vartheta_{x_0} \subset \gamma \setminus \gamma_N = \gamma_D$, and ϑ_{x_0} being an open set, $\vartheta_{x_0} \subset \overset{\circ}{\gamma}_D$. This not possible since θ_n vanishes on $\overset{\circ}{\gamma}_D$.

The "interval" $\vartheta_{x_0} =]a, b[$ (with respect to the curvilinear abcissa) can be chosen maximal, i.e. such that $\theta_n u_N^0(a) = \theta_n u_N^0(b) = 0$. Moreover, θ_n remains of constant sign (say positive) on ϑ_{x_0}. The point is now to prove that ∇u_N^0 vanishes as well on ϑ_{x_0}, which by Holmgren's theorem would lead to $u_N^0 \equiv 0$ in Ω.

We are now going to construct a special family of functions v, fulfilling the conditions of lemma 5, in order to achieve the proof of the theorem. Up to a local map, the "interval" ϑ_{x_0} can be identified to $]0,1[$. Then, given two positive real numbers c and ε, it is possible to construct a family of functions $\xi_\varepsilon \in C^2(\overline{\Omega})$ such that :

$$\begin{cases} \xi_\varepsilon \equiv 1 \text{ for } x \in]\varepsilon, 1-\varepsilon[\\ \xi_\varepsilon \equiv 0 \text{ for } x \in]0, \frac{\varepsilon}{2}[\cup]1-\frac{\varepsilon}{2}, 1[\\ 0 \leq \xi_\varepsilon \leq 1 \\ |\nabla \xi_\varepsilon(x)| \leq \dfrac{c}{\varepsilon} \quad \forall x \in \overline{\Omega} \\ \left|\nabla^2 \xi_\varepsilon(x)\right| \leq \dfrac{c}{\varepsilon^2} \quad \forall x \in \overline{\Omega} \end{cases} \tag{35}$$

Let us now denote ϑ_{x_0} by ϑ, and by ϑ_ε the "interval' $]\varepsilon, 1-\varepsilon[$. Using (28) with $v = \xi_\varepsilon u_N^0$ gives, for any $\varepsilon > 0$:

$$\int_{\vartheta_\varepsilon} \theta_n \left(\frac{\partial u_N^0}{\partial \tau}\right)^2 + \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} + \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n \left(\frac{\partial u_N^0}{\partial \tau}\right)^2 \xi_\varepsilon = 0 \tag{36}$$

Two situations have to be considered.

<u>First situation</u> : $\overline{\vartheta} \subset \gamma$.

Referring again to Khodja-Moussaoui [12], we know that $u_N^0 \in \mathcal{C}^{1,\beta}(\overline{\vartheta})$, $\beta < \dfrac{1}{2}$. There exist then two positive constants c_1 and c_2 such that :

$$\left| \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} \right| \leq \frac{c_1}{\varepsilon} \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n u_N^0 \tag{37}$$

and that

$$\left| \int_{\vartheta\backslash\vartheta_\varepsilon} \theta_n \left(\frac{\partial u_N^0}{\partial \tau} \right)^2 \xi_\varepsilon \right| \le c_2 \int_{\vartheta\backslash\vartheta_\varepsilon} \theta_n \tag{38}$$

It is clear that $\lim\limits_{\varepsilon\to 0} \int_{\vartheta\backslash\vartheta_\varepsilon} \theta_n = 0$.

As for the right handside of (37), let us denote by χ the local map $]0,1[\longmapsto \vartheta =]a,b[$. The mean value theorem gives us two real numbers $\alpha_\varepsilon \in]0,\varepsilon[$, and $\beta_\varepsilon \in]1-\varepsilon, 1[$ such that :

$$\int_{\vartheta\backslash\vartheta_\varepsilon} \theta_n\, u_N^0 = \left\{ \left[(\theta_n u_N^0)(\chi(\alpha_\varepsilon)\chi'(\alpha_\varepsilon) \right] + \left[(\theta_n u_N^0)(\chi(\beta_\varepsilon)\chi'(\beta_\varepsilon) \right] \right\} \varepsilon$$

and accordingly :

$$\left| \frac{1}{\varepsilon} \int_{\vartheta\backslash\vartheta_\varepsilon} \theta_n\, u_N^0 \right| = \left\{ \left[(\theta_n u_N^0)(\chi(\alpha_\varepsilon)\chi'(\alpha_\varepsilon) \right] + \left[(\theta_n u_N^0)(\chi(\beta_\varepsilon)\chi'(\beta_\varepsilon) \right] \right\}$$

It comes out then, since $(\theta_n u_N^0)$ is continuous on γ. and that $(\theta_n u_N^0)(\chi(0)) = (\theta_n u_N^0)(\chi(1)) = 0$:

$$\lim_{\varepsilon\to 0} \left| \frac{1}{\varepsilon} \int_{\vartheta\backslash\vartheta_\varepsilon} \theta_n\, u_N^0 \right| = 0$$

On the other hand, according to [12], $u_N^0 \in \mathcal{C}^{1,\beta}(\overline{\vartheta}))$, and we we get :

$$\lim_{\varepsilon\to 0} \int_{\vartheta_\varepsilon} \theta_n \left(\frac{\partial u_N^0}{\partial \tau} \right)^2 = 0$$

Making $\varepsilon \to 0$ in equation (36) then leads to :

$$\int_{\vartheta} \theta_n \left(\frac{\partial u_N^0}{\partial \tau} \right)^2 = 0$$

But $\theta_n > 0$ on ϑ, so that $\dfrac{\partial u_N^0}{\partial \tau} = 0$ on ϑ. Since $\dfrac{\partial u_N^0}{\partial n}$ is also vanishing on ϑ, the Holmgren's theorem provides the argument to conclude.

<u>Second situation</u> : $\overline{\vartheta} \not\subset \gamma$. We can suppose as well that $\vartheta = \gamma$.

In this situation, the regularity results are helpless, since they do not hold up to $\partial\gamma$. To prove the following

$$\lim_{\varepsilon\to 0} \int_{\vartheta-\vartheta_\varepsilon} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} = 0$$

it is sufficient to prove that the integral on the "interval" $\chi([0,\varepsilon])$ vanishes. Integrating by parts, we get :

$$\int_{\chi([0,\varepsilon])} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} = \int_{\chi([0,\varepsilon])} \theta_n u_N^0 \frac{\partial}{\partial \tau} \left(u_N^0\right)^2 \frac{\partial \xi_\varepsilon}{\partial \tau} = \left[\theta_n (u_N^0)^2 \frac{\partial \xi_\varepsilon}{\partial \tau} \right]_{\chi(0)}^{\chi(\varepsilon)} - \int_{\chi([0,\varepsilon])} \left(u_N^0\right)^2 \frac{\partial}{\partial \tau} \left(\theta_n \frac{\partial \xi_\varepsilon}{\partial \tau} \right) \tag{39}$$

The assumption on $\theta_n'(\chi(0)) = 0$ is used here to get $\theta_n(\chi(\varepsilon)) = \varepsilon o(\varepsilon)\,;\ \lim_{\varepsilon\to 0} o(\varepsilon) = 0$. On the other hand, u_N^0 is continuous at $\chi(0)$, and $|\nabla\xi_\varepsilon| \leq \frac{c}{\varepsilon}$, so that :

$$\lim_{\varepsilon\to 0}\left(\theta_n (u_N^0)^2 \frac{\partial \xi_\varepsilon}{\partial\tau}\right)(\chi(\varepsilon)) = 0 \tag{40}$$

Making use again - after the change of variables χ - of the mean value theorem on the interval $[0,\varepsilon]$, we can write down the following, for some $\alpha_\varepsilon \in]0,\varepsilon[$:

$$\int_{\chi([0,\varepsilon])} (u_N^0)^2 \frac{\partial}{\partial\tau}\left(\theta_n \frac{\partial\xi_\varepsilon}{\partial\tau}\right) = u_N^0(\chi(\alpha_\varepsilon))^2\chi'(\alpha_\varepsilon)\left[\theta_n(\alpha_\varepsilon)\frac{\partial^2\xi_\varepsilon}{\partial\tau^2}(\chi(\alpha_\varepsilon)) + \frac{\partial^2\xi_\varepsilon}{\partial\tau^2}(\chi(\alpha_\varepsilon))\frac{\partial\theta_n}{\partial\tau}(\chi(\alpha_\varepsilon))\right] \times \varepsilon$$

Using the following properties,

$$\left\{\begin{array}{l} u_N^0 \circ \chi \text{ and } \theta_n \circ \chi \text{ are continuous at } 0 \\ \left|\nabla^2 \xi_\varepsilon\right| \leq \dfrac{c}{\varepsilon^2} \\ \theta_n(\chi(\varepsilon)) = \varepsilon\, o_1(\varepsilon)\,;\ \lim\limits_{\varepsilon\to 0} o_1(\varepsilon) = 0 \\ \dfrac{\partial\theta_n}{\partial\tau}(\chi(\varepsilon)) = o_2(\varepsilon)\,;\ \lim\limits_{\varepsilon\to 0} o_2(\varepsilon) = 0 \\ |\nabla\xi_\varepsilon| \leq \dfrac{c}{\varepsilon} \end{array}\right. \tag{41}$$

we derive :

$$\lim_{\varepsilon\to 0}\int_{\chi([0,\varepsilon])} u_N^0 \frac{\partial}{\partial\tau}\left(\theta_n\frac{\partial\xi_\varepsilon}{\partial\tau}\right) = 0$$

which gives, by using (39) and (40) :

$$\lim_{\varepsilon\to 0}\int_{\chi([0,\varepsilon])} \theta_n u_N^0 \frac{\partial u_N^0}{\partial\tau}\frac{\partial\xi_\varepsilon}{\partial\tau} = 0$$

Of course, we have also $\lim\limits_{\varepsilon\to 0}\int_{\chi([1-\varepsilon,1])} \theta_n u_N^0 \frac{\partial u_N^0}{\partial\tau}\frac{\partial\xi_\varepsilon}{\partial\tau} = 0$, so that :

$$\lim_{\varepsilon\to 0}\int_{\vartheta_\varepsilon} \theta_n\left(\frac{\partial u_N^0}{\partial\tau}\right)^2 = 0$$

It follows, since $\theta_n > 0$ on ϑ, that $\frac{\partial u_N^0}{\partial\tau}$ vanishes on ϑ and, according to the fact that $\frac{\partial u_N^0}{\partial n}$ is also vanishing on ϑ, the Holmgren's theorem gives the final argument to conclude, which ends the proof of the theorem.

■

4 Conclusion

The inverse problem with unilateral boundary conditions for the Laplace equation is clearly not of great physical interest. However, most the theoretical difficulties expected in more realistic situations (namely the inverse elastic problem, or the coupled thermoelastic one), are as well gathered in the present "model" problem, which makes its study of great interest.

The identifiability uses classical tools : the Holmgren's continuation theorem, and variational arguments. As for Lipschitz stability results, they are also based on the Holmgren's theorem, and use as a basic tool the derivatives with respect to the domain. However, serious difficulties arise from the possible lack of connectivity of parts (γ_D, γ_N) the unknown boundary defined by the Signorini solution :although this latter is smoother than the solution of the related mixed linear problem, the possibility that $(\gamma_D \setminus \overset{\circ}{\gamma}_D)$ be some closed set of positive measure and void interior, such as a Cantor p-adic set, could not be excluded. In such a situation, the Holmgren's theorem is no more the "magic" straightforward tool we are used to in the linear situations. The conditions for its final use have to be patiently built up, by using sharp informations on the structure of the Signorini solution on the unknown boundary, backed with arguments coming from the analytical functions theory. The Lipschitz stability result proved this way is hence limited to 2D situations, although an extension to 3D might be not excluded.

The development of an appropriate identification algorithm, which is the aim of a forthcoming work currently in progress, will also be facing difficulties similar to those encountered above, particularly when differentiating the cost function. The present work provides useful tools to overcome them.

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